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Continuity of solution maps of parametric quasiequilibrium problems

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Abstract In this article, we consider a parametric vector quasiequilibrium problem in topological vector spaces. Sufficient conditions for solution maps to be lower and Hausdorff lower semicontinuous, upper semicontinuous and continuous are established. Our results improve recent existing ones in the literature.

Keywords Parametric vector quasiequilibrium problems · Solution maps · Lower semicontinuity · Hausdorff lower semicontinuity · Upper semicontinuity · Continuity · Generalized concavity · Generalized pseudomonotonicity

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1 Introduction

Throughout the paper, unless otherwise specified, let X, Y and Λ be Hausdorff topological vector spaces. Let $A \subseteq X$ be nonempty. Let $K : A \times \Lambda \to 2^A$, $\Gamma : A \times \Lambda \to 2^Y$ be multifunctions and $f : A \times A \times \Lambda \to Y$ be a mapping. Assume that the values of Γ are closed with nonempty interiors different from Y. For $\lambda \in \Lambda$ consider the following parametric quasiequilibrium problem
(QEP) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that, for all $y \in K(\bar{x}, \lambda)$,

$$f(\bar{x}, y, \lambda) \in \Gamma(\bar{x}, \lambda).$$

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This problem setting encompasses many important problems as special cases, see Sect. 5. Semicontinuity and continuity properties of solutions have been investigated even in more general models like inclusion problems [7,10], variational relation problems [16], or in more general settings with set-valued mappings for quasiequilibrium problems [2,5,8,9]. However, to get stability properties, assumptions imposed for such general models may become too restrictive when applied to the above problem (QEP). Semicontinuity conditions for multivalued mapping f become strong continuity requirements for single-valued f. While some (generalized) convexity assumptions do not have counterparts for the single-valued case. This observation motivates the aim of this paper, which is to establish sufficient conditions for solution sets of the above problem (QEP) to be semicontinuous or continuous with respect to parameter λ under relaxed assumptions about continuity-related and generalized convexity of the (single-valued) mapping f. Several of our new assumptions may look rather technical but are in fact easy to be checked, and what is more important, easier to be fulfilled in practical problems. Our results improve the corresponding ones of [13,14,17–20] when applied to the particular cases studied in these works. Note that parametric problems in the literature often involve several parameters (perturbing constraints and f independently). However, all these parameters can be considered as one of a suitable defined product space. Hence we include only one parameter in our problem setting.

The structure of our paper is as follows. In the remaining part of this section we recall definitions for later uses. Section 2 is devoted to upper semicontinuity while Sect. 3 deals with lower and Hausdorff lower semicontinuities. Conditions for continuity of solution maps of (QEP) are provided in the next Sect. 4. The last Sect. 5 concerns several particular cases as examples, where we derive consequences of our main results.

The following definitions for set-valued maps can be seen in, e.g., [12]. For topological spaces X, Y and set-valued mapping $Q : X \to 2^Y$, recall that Q is called upper semicontinuous (usc in short; lower semicontinuous, lsc, respectively) at x_0 if for open subset U of Y with $Q(x_0) \subseteq U$ ($Q(x_0) \cap U \neq \emptyset$), there is a neighborhood N of x_0 such that $Q(N) \subseteq U$ ($\forall x \in N, Q(x) \cap U \neq \emptyset$). An equivalent formulation for lower semicontinuity is that: Q is lsc at x_0 if, for all $x_\alpha \to x_0$ and $y \in Q(x_0)$, there exists $y_\alpha \in Q(x_\alpha)$ such that $y_\alpha \to y$. Since we largely use this equivalent statement in the sequel, as a referee suggested we give a direct proof here. Let $x_\alpha \to x_0$ and $y \in Q(x_0)$. For any open neighborhood U of y, as $Q(x_0) \cap U \neq \emptyset$, there is a neighborhood N of x_0 such that, $\forall x \in N$, $Q(x) \cap U \neq \emptyset$. Take $x_\alpha \in N$ and $y_\alpha \in Q(x_\alpha) \cap U$. Since U is arbitrary, we can do this such that $y_\alpha \to y$. For the converse, suppose ad absurdum the existence of an open subset U with $Q(x_0) \cap U \neq \emptyset$ such that each neighborhood N of x_0 contains a point x_N with $Q(x_N) \cap U = \emptyset$. We can choose such neighborhoods N such that the corresponding x_N form a net converging to x_0. By the assumption, there is a corresponding net $\{y_N\}$ with $y_N \in Q(x_N)$ and $y_N \to y$, which contradicts the fact $Q(x_N) \cap U = \emptyset$.

Q is said to be continuous at x_0 if it is both lsc and usc at x_0. When Y is a topological vector space, Q is said to be Hausdorff upper semicontinuous (H-usc in short; Hausdorff lower semicontinuous, H-lsc, respectively) at x_0 if, for each neighborhood B of the origin in Y, there exists a neighborhood N of x_0 such that, $Q(x) \subseteq Q(x_0) + B, \forall x \in N$ ($Q(x_0) \subseteq Q(x) + B, \forall x \in N$). Q is termed closed at x_0 if, for any net $\{(x_\alpha, y_\alpha)\} \subseteq \operatorname{graph} Q := \{(x, y) \in X \times Y \mid y \in Q(x)\}$ with $(x_\alpha, y_\alpha) \to (x_0, y_0)$, we have $y_0 \in Q(x_0)$. We say that Q satisfies a certain property in a subset $A \subseteq X$ if Q satisfies it at every point of A. If $A = \operatorname{dom} Q := \{x \mid Q(x) \neq \emptyset\}$ we omit "in dom$Q$" in the saying.

For a set-valued map $Q : X \to 2^Y$ between two linear spaces, Q is said to be concave on a convex subset $A \subseteq X$ if, for each $x_1, x_2 \in A$ and $t \in [0, 1]$,

$$Q((1-t)x_1 + tx_2) \subseteq tQ(x_1) + (1-t)Q(x_2).$$

Denote $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_- = -\mathbb{R}_+$. For a subset A of X, int$A$, cl$A$ and bd$A$ stand for the interior, closure and boundary of A, respectively. For X, Y, Γ, F as in (QEP) and $\theta \in Y$, we will use the following level-set types:

$$\text{lev}_{\theta.\Gamma} := \{(x, y, \lambda) \mid f(x, y, \lambda) \in \theta + \Gamma(x, \lambda)\},$$
$$\text{lev}_{\theta.\Gamma(.,\lambda_0)} := \{(x, y) \mid f(x, y, \lambda_0) \in \theta + \Gamma(x, \lambda_0)\}.$$

2 Upper semicontinuity of solution maps

In this section, we discuss upper semicontinuity of solution maps to our parametric quasi-equilibrium problem (QEP) under relaxed conditions. To this end, we propose generalized convexity and monotonicity properties as follows.

Definition 2.1 Let $g : X \times Z \to Y$, $\Delta : X \times Z \to 2^Y$, where Δ has values with nonempty interiors. g is called generalized Δ-concave (with respect to the second variable) in a convex set $A \subseteq Z$, if for each $x \in X$ and $z_1, z_2 \in A$, from $g(x, z_1) \in \Delta(x, z_1)$ and $g(x, z_2) \in \text{int}\,\Delta(x, z_2)$, it follows that, for all $t \in (0, 1)$,

$$g(x, (1-t)z_1 + tz_2) \in \text{int}\,\Delta(x, (1-t)x_1 + tz_2).$$

Definition 2.2 Let $g : X \times X \to Y$ be a function and $\Delta : X \to 2^Y$ be a multifunction with the values having nonempty interiors.

(i) g is called Δ-quasimonotone in $A \subseteq X$ if, for all $x \neq y$ in A,

$$[g(x, y) \in \text{int}\,\Delta(x)] \Longrightarrow [g(y, x) \notin \text{int}\,\Delta(y)].$$

(ii) g is termed Δ-pseudomonotone in $A \subseteq X$ if, for all $x \neq y$ in A,

$$[g(x, y) \in \Delta(x)] \Longrightarrow [g(y, x) \notin \text{int}\,\Delta(y)].$$

In dealing with particular cases of (QEP) in Sect. 5, we need the following classical definition, which is obtained from Definition 2.2 by setting $Y = \mathbb{R}$, $\Delta(x) \equiv \mathbb{R}_+$ and $g(x, y) = \langle b(x), y - x\rangle$.

Definition 2.3 (See e.g., [23]) Let X be a normed space, $A \subseteq X$ be a nonempty subset and $b : A \to X^*$ be a mapping.

(a) b is said to be quasimonotone in A, if for each x, y in A,

$$[\langle b(x), y - x\rangle > 0] \Longrightarrow [\langle b(y), x - y\rangle \leq 0].$$

(b) b is said to be pseudomonotone in A, if for each x, y in A,

$$[\langle b(x), y - x\rangle \geq 0] \Longrightarrow [\langle b(y), x - y\rangle \leq 0].$$

In the sequel let, for $\lambda \in \Lambda$,

$$E(\lambda) = \{x \in A \mid x \in K(x, \lambda)\}$$

and $S(\lambda)$ be the solution set of problem (QEP) corresponding to λ. Since the existence of solutions for (QEP) has been intensively studied in the literature, we focus on the stability study, assuming always that $S(\lambda) \neq \emptyset$.

Theorem 2.1 *For problem* (QEP) *assume that*

(*i*) *E is usc at λ_0, $E(\lambda_0)$ is compact and K is lsc in $A \times \Lambda$;*
(*ii*) $\text{lev}_{0.\Gamma(.,\lambda_0)} f(.,.,\lambda_0)$ *is closed in $K(A,\Lambda) \times K(A,\Lambda)$;*
(*iii*) *for all x, y in $K(A,\Lambda)$, $f(x,y,.)$ is $Y\backslash\Gamma(x,.)$- usc at λ_0, uniformly with respect to x, y in the sense that, if $f(x,y,\lambda_0) \in Y\backslash\Gamma(x,\lambda_0)$, there is a neighborhood N of λ_0, not depending on x, y, such that*

$$f(x,y,\lambda) \subseteq Y\backslash\Gamma(x,\lambda), \quad \forall\lambda \in N.$$

Then the solution map S is usc at λ_0.

Proof Suppose that S is not usc at λ_0, i.e., there is an open superset U of $S(\lambda_0)$ such that there are nets $\lambda_\alpha \to \lambda_0$ and $x_\alpha \in S(\lambda_\alpha)$ with $x_\alpha \notin U$ for all α. By the upper semicontinuity of E and the compactness of $E(\lambda_0)$ one can assume that $x_\alpha \to x_0$, for some $x_0 \in E(\lambda_0)$. Suppose there is $y_0 \in K(x_0,\lambda_0)$ such that $f(x_0,y_0,\lambda_0) \in Y\backslash\Gamma(x_0,\lambda_0)$. The lower semicontinuity of K in turn shows the existence of $y_\alpha \in K(x_\alpha,\lambda_\alpha)$ such that $y_\alpha \to y_0$. Condition (ii) allows one to assume that

$$f(x_\alpha,y_\alpha,\lambda_0) \in Y\backslash\Gamma(x_\alpha,\lambda_0).$$

Since $f(x,y,.)$ is $Y\backslash\Gamma(x,.)$-usc at λ_0, there is a neighborhood N of λ_0 such that

$$f(x_\alpha,y_\alpha,\lambda) \subseteq Y\backslash\Gamma(x_\alpha,\lambda), \quad \forall\lambda \in N,$$

which is impossible as $x_\alpha \in S(\lambda_\alpha)$ for all α. Thus, $x_0 \in S(\lambda_0) \subseteq U$, which is again a contradiction, since $x_\alpha \notin U$ for all α. □

Remark 2.1 When $K(x,\lambda) \equiv K$ and $\Gamma(x,\lambda) \equiv \Gamma$, it is not hard to check that the closedness assumption (ii) for $f(.,.,\lambda_0)$ can be relaxed to that for $f(.,y,\lambda_0)$, for all $y \in K$, and the uniformity with respect to $x, y \in K$ in (iii) can be weakened to the uniformity with respect to $x \in K$. Therefore, Theorem 2.1 improves Theorem 3.1 of [13] and Theorem 2.1 of [14], since our assumptions are imposed only in K (not globally in A as in the mentioned theorems) and our semicontinuity assumption in (iii) is weaker than the counterpart in these theorems.

Assumption (iii) of Theorem 2.1 is essential as shown by the following example.

Example 2.1 Let $X = A = Y = l_2$, $\Lambda = [0,1]$, $\Gamma(x,\lambda) = \{y \in l_2 \mid y_k \geq 0, k = 1,2,\ldots\}$, $K(x,\lambda) = \left\{y \in l_2 \mid 0 \leq y_n \leq \frac{1}{n}\right\}$, $\lambda_0 = 0$ and

$$f(x,y,\lambda) = \begin{cases} x - y, & \text{if } \lambda = 0, \\ (x_1(x_1 - y_1), x_2(x_2 - y_2), \ldots), & \text{otherwise,} \end{cases}$$

where $l_2 = \{x = (x_1, x_2, \ldots) \mid \sum_{n=1}^{\infty} x_n^2 < +\infty\}$. Then (i) is satisfied as $K(x,\lambda)$ is constant and compact. (ii) is fulfilled since $f(.,.,0)$ is continuous. It is clear that $S(0) = \left\{\left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right)\right\}$ and $S(\lambda) = \left\{(0,0,\ldots), \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right)\right\}$ and hence S is not usc at 0. The reason is that assumption (iii) is violated. Indeed, taking $x = (0,0,\ldots)$, $y = \left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2n}, \ldots\right)$, one has, for $\lambda \neq 0$,

$$f(x,y,0) = \left(-\frac{1}{2}, -\frac{1}{4}, \ldots, -\frac{1}{2n}, \ldots\right) \in l_2\backslash\Gamma,$$
$$f(x,y,\lambda) = (0,0,\ldots) \notin l_2\backslash\Gamma.$$

Although assumption (iii) cannot be dispensed within the statement, we can replace it as follows.

Theorem 2.2 *Assume that*

(i) *E is usc at λ_0, $E(\lambda_0)$ is compact and K is lsc in $A \times \Lambda$;*
(ii') $\text{lev}_{0.\Gamma} f$ *is closed in $K(A, \Lambda) \times K(A, \Lambda) \times \{\lambda_0\}$.*

Then the solution map S of (QEP) is usc at λ_0.

Proof Reasoning ad absurdum, suppose the existence of an open subset $U \supseteq S(\lambda_0)$ and a net $\{(x_\alpha, \lambda_\alpha)\}$ converging to (x_0, λ_0) such that $x_\alpha \in S(\lambda_\alpha)\backslash U$ for all α. If $x_0 \notin S(\lambda_0)$, there is $y_0 \in K(x_0, \lambda_0)$ with $f(x_0, y_0, \lambda_0) \in Y\backslash\Gamma(x_0, \lambda_0)$. Since K is lsc at (x_0, λ_0), there exists a net $\{y_\alpha\}$ with $y_\alpha \in K(x_\alpha, \lambda_\alpha)$ and $y_\alpha \to y_0$. As $x_\alpha \in S(\lambda_\alpha)$, $f(x_\alpha, y_\alpha, \lambda_\alpha) \in \Gamma(x_\alpha, \lambda_\alpha)$. From assumption (ii′) we have $f(x_0, y_0, \lambda_0) \in \Gamma(x_0, \lambda_0)$, a contradiction. If $x_0 \in S(\lambda_0) \subseteq U$, one has another contradiction, as $x_\alpha \notin U$ for all α. □

Remark 2.2 For the special case of our (QEP), where $\Gamma(x, \lambda) = Y\backslash - \text{int}C(x, \lambda)$, $C(x, \lambda)$ being a convex cone, Theorem 4.1 of [19] has the same conclusion as that of Theorem 2.2. Assumption (v) in this theorem coincides with (ii′) in Theorem 2.2. It is assumed in (i)–(iv) of this theorem that A is compact, K is usc and compact-valued. These assumptions are easily seen to imply our assumptions that E is usc at λ_0 and $E(\lambda_0)$ is compact. So with the additional condition that K is lsc in $A \times \Lambda$ (which is not imposed in this Theorem 4.1), the assumptions of this theorem are stronger than those of our Theorem 2.2. The following example demonstrates that the lower semicontinuity of K needs to be added to Theorem 4.1 of [19].

Example 2.2 Let $X = Y = \mathbb{R}$, $A = [-1, 1]$, $\Lambda = [0, 1]$, $\Gamma(x, \lambda) = \mathbb{R}_+$,$\lambda_0 = 0$,

$$K(x, \lambda) = \begin{cases} \{-1, 0, 1\}, & \text{if} \quad \lambda = 0, \\ \{0, 1\}, & \text{otherwise,} \end{cases}$$

and $f(x, y, \lambda) = x + y + \lambda$. It is clear that $E(.) = K(.)$ is usc at 0 (K does not depend on x), $E(0)$ is compact. Condition (ii′) holds since f is continuous. The assumptions of Theorem 4.1 of [19] are easily seen to be fulfilled. But $S(0) = \{1\}$, $S(\lambda) = \{0, 1\}$, $\forall\lambda \in (0, 1]$ and so S is not usc at 0. The reason is that K is not lsc. (As a referee required, we note that the argument in [19] that, for arbitrary open neighborhood U of $y_0 \in K(x_0, \lambda_0)$, one gets $y_\alpha \in K(x_\alpha, \lambda_\alpha)$ with $(x_\alpha, \lambda_\alpha) \to (x_0, \lambda_0)$ such that $y_\alpha \in U$, is not adequate if K is not lsc at (x_0, λ_0).)

The following example shows a case where the assumed compactness in Theorem 4.1 of [19] is violated but the assumptions of Theorem 2.2 are fulfilled, and in fact S is usc.

Example 2.3 Let $X = Y = \mathbb{R}$, $\Lambda = [0, 1]$, $A = [0, 2)$, $\Gamma(x, \lambda) = \mathbb{R}_+$, $K(x, \lambda) = (x - \lambda - 1, \lambda] \cap A$, $\lambda_0 = 0$ and $f(x, y, \lambda) = x - y$. Then it is evident that the assumptions of Theorem 2.2 are fulfilled. A direct calculation gives $S(\lambda) = \{\lambda\}$, which is usc, although A and the values of K are not compact.

We can modify the closedness assumptions of Theorems 2.1 and 2.2, using generalized pseudomonotonicity and concavity as follows.

Theorem 2.3 *For (QEP) let $K(x, \lambda_0)$ be convex for $x \in A$ and the following conditions hold.*

(i) *E is usc at λ_0, $E(\lambda_0)$ is compact and K is lsc in $A \times \Lambda$.*
(ii') $\text{lev}_{0.(Y\backslash\text{int}\Gamma)} f$ *is closed in $K(A, \Lambda) \times K(A, \Lambda) \times \{\lambda_0\}$ and* $\text{lev}_{0.\Gamma(.,\lambda_0)} f(., y, \lambda_0)$ *is closed for all $y \in K(A, \lambda_0)$.*

(*iii*) *There is a neighborhood* U *of* λ_0 *such that,* $\forall\lambda \in U(\lambda_0)$*,* $f(.,.,\lambda)$ *is* $\Gamma(.,\lambda)$*-pseudomonotone in* $K(A,\lambda) \times K(A,\lambda)$*.*

(*iv*) *For each* $x \in K(A,\lambda_0)$*,* $f(x,.,\lambda_0)$ *is generalized* $Y\backslash\text{int}\Gamma(x,\lambda_0)$*-concave in* $K(x,\lambda_0)$ *and* $f(x,x,\lambda_0) \in \Gamma(x,\lambda_0)$*.*

Then the solution map S *is usc at* λ_0*.*

Proof We first prove that S is closed at λ_0 by considering $\lambda_\alpha \to \lambda_0$ and $x_\alpha \in S(\lambda_\alpha)$ with $x_\alpha \to x_0$. For each $y \in K(x_0,\lambda_0)$, $y_\alpha \in K(x_\alpha,\lambda_\alpha)$ exists such that $y_\alpha \to y$, since K is lsc at (x_0,λ_0). By the pseudomonotonicity of $f(.,.,\lambda_\alpha)$ one has

$$f(y_\alpha, x_\alpha, \lambda_\alpha) \in Y\backslash\text{int}\Gamma(y_\alpha, \lambda_\alpha).$$

The closedness of $\text{lev}_{0.(Y\backslash\text{int}\Gamma)} f$ implies that

$$f(y, x_0, \lambda_0) \in Y\backslash\text{int}\Gamma(y, \lambda_0). \tag{1}$$

We claim that $f(x_0,y_0,\lambda_0) \in \Gamma(x_0,\lambda_0)$ for each $y_0 \in K(x_0,\lambda_0)$. As $x_0 \in K(x_0,\lambda_0)$, one has $y_t = (1-t)x_0 + ty_0 \in K(x_0,\lambda_0)$, for $t \in (0,1)$. Suppose $f(y_t,y_0,\lambda_0) \in Y\backslash\Gamma(y_t,\lambda_0)$. If $f(y_t,x_0,\lambda_0) \in Y\backslash\Gamma(y_t,\lambda_0)$, from the assumed $Y\backslash\text{int}\Gamma(y_t,\lambda_0)$-concavity, one gets $f(y_t,y_t,\lambda_0) \in Y\backslash\Gamma(y_t,\lambda_0)$, which is impossible. While if $f(y_t,x_0,\lambda_0) \in \Gamma(y_t,\lambda_0)$, (1) implies that $f(y_t,x_0,\lambda_0) \in \text{bd}\Gamma(y_t,\lambda_0) = \text{bd}(Y\backslash\text{int}\Gamma(y_t,\lambda_0)) \subseteq Y\backslash\text{int}\Gamma(y_t,\lambda_0)$. The generalized $Y\backslash\text{int}\Gamma(y_t,\lambda_0)$-concavity yields $f(y_t,y_t,\lambda_0) \in Y\backslash\Gamma(y_t,\lambda_0)$, a contradiction. So $f(y_t,y_0,\lambda_0) \in \Gamma(y_t,\lambda_0)$. Passing to the limit as $t \to 0^+$ yields $f(x_0,y_0,\lambda_0) \in \Gamma(x_0,\lambda_0)$, since $\text{lev}_{0.\Gamma(.,\lambda_0)} f(.,y_0,\lambda_0)$ is closed. Hence $x_0 \in S(\lambda_0)$ and then S is closed at λ_0.

Now we show that S is usc at λ_0. Suppose an open superset U of $S(\lambda_0)$ exists such that there are nets $\{\lambda_\alpha\}$ converging to λ_0 and $x_\alpha \in S(\lambda_\alpha)$ with $x_\alpha \notin U$ for all α. By the upper semicontinuity of E and the compactness of $E(\lambda_0)$ one can assume that $x_\alpha \to x_0$, for some $x_0 \in E(\lambda_0)$. Since S is closed at λ_0, we have $x_0 \in S(\lambda_0) \subseteq U$, which is impossible since $x_\alpha \notin U$ for all α. □

The level set $\text{lev}_{0.(Y\backslash\text{int}\Gamma)} f$ is "opposite" to the other level sets involved in Theorems 2.1–2.3. The following example ensures that its assumed closedness cannot be dropped.

Example 2.4 Let $X = Y = A = \mathbb{R}$, $\Lambda = [0,1]$, $\Gamma(x,\lambda) = \mathbb{R}_+$, $K(x,\lambda) = [0,1]$, $\lambda_0 = 0$ and

$$f(x,y,\lambda) = \begin{cases} x - y, & \text{if } \lambda = 0, \\ xy(x-y), & \text{otherwise.} \end{cases}$$

Then assumption (i) is satisfied. Since $f(x,y,0) = x - y$, $\text{lev}_{0.\mathbb{R}_+} f(.,y,0)$ is closed. It is easy to see that $f(.,.,\lambda)$ is $\mathbb{R}_+$ pseudomonotone in $[0,1] \times [0,1]$ and hence (iii) holds. We check (iv). If $f(x,y,0) \le 0$ and $f(x,z,0) < 0$, then $x \le y$ and $x < z$. So $f(x,(1-t)y + tz, 0) = x - (1-t)y - tz < 0$, for all $t \in (0,1)$, i.e., $f(x,.,0)$ is generalized $\mathbb{R}_-$-concave. The assumptions of Theorem 2.3 are fulfilled except the closedness of $\text{lev}_{0.\mathbb{R}_-} f$. It is clear that $S(0) = \{1\}$, $S(\lambda) = \{0,1\}$, $\forall\lambda \in (0,1]$, and hence S is not usc at 0. The reason is that $\text{lev}_{0.\mathbb{R}_-} f$ is not closed. (To see this let $x_n = 1$, $y_n = 0$ and $\lambda_n = \frac{1}{n}$. Then $(x_n,y_n,\lambda_n) \to (1,0,0)$ and $f(x_n,y_n,\lambda_n) = 0$, but $f(1,0,0) = 1 > 0$.)

3 Lower semicontinuity of solution maps

For investigation of lower semicontinuity of solution maps to (QEP), as an auxiliary problem we consider the following one:

(QEP_1) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that, for all $y \in K(\bar{x}, \lambda)$,

$$f(\bar{x}, y, \lambda) \in \mathrm{int}\Gamma(\bar{x}, \lambda),$$

where X, Λ, A, K, Γ and f are as in Sect. 1. Let $S_1(\lambda)$ be the solution set of (QEP_1) corresponding to λ. Clearly $S_1(\lambda) \subseteq S(\lambda)$.

Theorem 3.1 *Assume that $S_1(\lambda) \neq \emptyset$ and that*

- (i) *E is lsc at λ_0 and $E(\lambda_0)$ is convex; K is usc and compact-valued in $E(\lambda_0) \times \{\lambda_0\}$; $K(., \lambda_0)$ is concave in $E(\lambda_0)$;*
- (ii) $\mathrm{lev}_{0.Y\setminus int\Gamma} f$ *is closed in $K(A, \Lambda) \times K(A, \Lambda) \times \{\lambda_0\}$;*
- (iii) *$f(., ., \lambda_0)$ is generalized $\Gamma(., \lambda_0)$-concave in $E(\lambda_0) \times K(A, \lambda_0)$.*

Then the solution map S of (QEP) is lsc at λ_0.

Proof We start by proving that S_1 is lsc at λ_0. Suppose to the contrary that S_1 is not lsc at λ_0, i.e., there are $x_0 \in S_1(\lambda_0)$ and net $\{\lambda_\alpha\} \subseteq \Lambda$ converging to λ_0 such that, for all $x_\alpha \in S_1(\lambda_\alpha)$, the net $\{x_\alpha\}$ does not converge to x_0. Since E is lsc at λ_0, there is $\bar{x}_\alpha \in E(\lambda_\alpha)$ with $\bar{x}_\alpha \to x_0$. By the above contradiction assumption, there must be a subnet $\{\bar{x}_\beta\}$ such that, for all β, $\bar{x}_\beta \notin S_1(\lambda_\beta)$, i.e., for some $y_\beta \in K(\bar{x}_\beta, \lambda_\beta)$,

$$f(\bar{x}_\beta, y_\beta, \lambda_\beta) \in Y\setminus\mathrm{int}\Gamma(\bar{x}_\beta, \lambda_\beta). \tag{2}$$

As K is usc at (x_0, λ_0) and $K(x_0, \lambda_0)$ is compact one has $y_0 \in K(x_0, \lambda_0)$ such that $y_\beta \to y_0$ (taking a subnet if necessary). By assumption (ii), (2) yields that $f(x_0, y_0, \lambda_0) \in Y\setminus\mathrm{int}\Gamma(x_0, \lambda_0)$, which is impossible since $x_0 \in S_1(\lambda_0)$.

Now let us check that

$$S(\lambda_0) \subseteq \mathrm{cl}S_1(\lambda_0). \tag{3}$$

Let $\bar{x} \in S(\lambda_0), \bar{x}^1 \in S_1(\lambda_0)$ and $x_t = (1-t)\bar{x} + t\bar{x}^1$ with $t \in (0, 1)$. Since $K(., \lambda_0)$ is concave, for all $y \in K(x_t, \lambda_0)$, there exist $\bar{y} \in K(\bar{x}, \lambda_0)$ and $\bar{y}_1 \in K(\bar{x}^1, \lambda_0)$ such that $y = (1-t)\bar{y} + t\bar{y}_1$. Since $f(., ., \lambda_0)$ is generalized $\Gamma(., \lambda_0)$-concave, $f(x_t, y, \lambda_0) \in \mathrm{int}\Gamma(x_t, \lambda_0)$, i.e., $x_t \in S_1(\lambda_0)$. Therefore, (3) holds. By the lower semicontinuity of S_1 at λ_0 we have

$$S(\lambda_0) \subseteq \mathrm{cl}S_1(\lambda_0) \subseteq \liminf S_1(\lambda_\alpha) \subseteq \liminf S(\lambda_\alpha),$$

i.e., S is lsc at λ_0. □

The following example makes it clear that the concavity of $f(., ., \lambda_0)$ is essential.

Example 3.1 Let $X = Y = A = \mathbb{R}$, $\Lambda = [0, 1]$, $\Gamma(x, \lambda) = \mathbb{R}_+$, $K(x, \lambda) = [\lambda, \lambda + 3]$, $\lambda_0 = 0$ and $f(x, y, \lambda) = x^2 - (\lambda + 1)x$. Then, it is easy to see that assumptions (i) and (ii) of Theorem 3.1 are satisfied. But $S(0) = \{0\} \cup [1, 3]$ and $S(\lambda) = [\lambda + 1, \lambda + 3], \forall\lambda \in (0, 1]$, and hence $S(.)$ is not lsc at 0. The reason is that (iii) is violated. Indeed, let $x_1 = 0, x_2 = \frac{3}{2} \in E(0) = [0, 3]$. Then, $\forall y \in K(A, 0) = [0, 3]$, we have $f(x_1, y, 0) = 0$, $f(x_2, y, 0) = \frac{3}{4}$, but

$$f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2, y, 0\right) = -\frac{3}{16}.$$

Remark 3.1 In Theorem 5.1 of [19] the same conclusion as Theorem 3.1 was proved in another way. Its assumptions (i)–(v) derive (i) of Theorem 3.1, assumptions (vi), (vii) coincide with (ii), (iii) of Theorem 3.1. Theorem 3.1 slightly improves Theorem 5.1 of [19], since no convexity of the values of K is imposed.

We now proceed to Hausdorff lower semicontinuity.

Theorem 3.2 *Impose the assumptions of Theorem 3.1 and the following additional conditions:*

(v) $K(., \lambda_0)$ *is lsc in* $E(\lambda_0)$ *and* $E(\lambda_0)$ *is compact*;
(vi) $\text{lev}_{0.\Gamma(.,\lambda_0)} f(., ., \lambda_0)$ *is closed in* $K(A, \Lambda) \times K(A, \Lambda)$.

Then S *is Hausdorff lower semicontinuous at* λ_0.

Proof We first show that $S(\lambda_0)$ is closed. Let $x_\alpha \in S(\lambda_0)$ $(\subseteq E(\lambda_0))$ be such that $x_\alpha \to x_0$ in $E(\lambda_0)$. Suppose there exists $y_0 \in K(x_0, \lambda_0)$ such that

$$f(x_0, y_0, \lambda_0) \in Y \backslash \Gamma(x_0, \lambda_0). \tag{4}$$

Since $K(., \lambda_0)$ is lsc at x_0, there is $y_\alpha \in K(x_\alpha, \lambda_0)$ with $y_\alpha \to y_0$. As $x_\alpha \in S(\lambda_0)$, we have

$$f(x_\alpha, y_\alpha, \lambda_0) \in \Gamma(x_\alpha, \lambda_0). \tag{5}$$

Assumption (vi) shows a contradiction between (4) and (5). Thus, $S(\lambda_0)$ is closed and then compact.

Now suppose S is not Hlsc at λ_0, i.e., there are a neighborhood B of the origin in X and $\lambda_\alpha \to \lambda_0$ such that, for all α, there exists $x_{0\alpha} \in S(\lambda_0) \backslash (S(\lambda_\alpha) + B)$. Since $S(\lambda_0)$ is compact, we can assume that $x_{0\alpha} \to x_0$ for some $x_0 \in S(\lambda_0)$. Then there are α_1, a neighborhood B_1 of 0 in X with $B_1 + B_1 \subseteq B$ and $b_\alpha \in B_1$ such that $x_{0\alpha} = x_0 + b_\alpha$, for all $\alpha \geq \alpha_1$. Since S is lsc at λ_0, there is $z_\alpha \in S(\lambda_\alpha)$ with $z_\alpha \to x_0$ and then there is α_2 such that, for each $\alpha \geq \alpha_2$, $b'_\alpha \in B_1$ exists with $z_\alpha = x_0 - b'_\alpha$. Consequently, for all $\alpha \geq \alpha_0 = \max\{\alpha_1, \alpha_2\}$,

$$x_{0\alpha} = x_0 + b_\alpha = z_\alpha + b'_\alpha + b_\alpha \in z_\alpha + B.$$

This is impossible due to the fact that $x_{0\alpha} \notin S(\lambda_\alpha) + B$. Thus, S is Hlsc at λ_0. □

The following example shows that the assumed compactness in (v) is essential.

Example 3.2 Let $X = A = \mathbb{R}^2, Y = \mathbb{R}, \Lambda = [0, 1], \Gamma(x, \lambda) = \mathbb{R}_+, \lambda_0 = 0$ and, for $x = (x_1, x_2) \in \mathbb{R}^2$, $K(x, \lambda) = \{(x_1, \lambda x_1)\}$ and $f(x, y, \lambda) = 1 + \lambda$. Then $E(\lambda) = \{(x_1, x_2) \mid x_2 = \lambda x_1\}$. Clearly the assumptions of Theorem 3.2, but the compactness of $E(\lambda_0)$, are satisfied. Direct computations give $S(\lambda) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = \lambda x_1\}$ and then S is not Hlsc at 0 (although S is lsc at 0).

4 Continuity of solution maps

We can combine the results in Sect. 2 and Theorem 3.1 to derive sufficient conditions for continuity of solution maps of (QEP). In this section we develop some conditions without concavity assumptions.

Theorem 4.1 *Impose the assumptions of either of Theorems 2.1–2.3. Assume further that*

(a) $f(., ., \lambda_0)$ *is* $\Gamma(., \lambda_0)$*-quasimonotone in* $K(A, \lambda_0) \times K(A, \lambda_0)$;
(b) *for each* $x \in S(\lambda_0)$ *and each* $y \in S(\lambda_0) \backslash \{x\}$, $f(x, y, \lambda_0) \in \text{int}\Gamma(x, \lambda_0)$.

Then S *is continuous at* λ_0.

Proof First let the assumptions of Theorem 2.1 (or 2.2) be satisfied. It suffices to prove that S is lsc at λ_0. Suppose to the contrary that there are a net $\{\lambda_\alpha\}$ converging to λ_0 and $x_0 \in S(\lambda_0)$ such that $x_\alpha \in S(\lambda_\alpha)$, but the net $\{x_\alpha\}$ does not converge to x_0. Since E is usc and $E(\lambda_0)$ is compact, we can assume that $x_\alpha \to \bar{x}_0$ for some $\bar{x}_0 \in E(\lambda_0)$. From the proof of Theorem 2.1 (or 2.2), we see that $\bar{x}_0 \in S(\lambda_0)$. By the contradiction assumption we have $\bar{x}_0 \neq x_0$. Due to assumption (b) one has

$$f(\bar{x}_0, x_0, \lambda_0) \in \mathrm{int}\Gamma(\bar{x}_0, \lambda_0) \quad \text{and} \quad f(x_0, \bar{x}_0, \lambda_0) \in \mathrm{int}\Gamma(x_0, \lambda_0),$$

which is impossible since $f(., ., \lambda_0)$ is $\Gamma(., \lambda_0)$-quasimonotone.

The case, where the assumptions of Theorem 2.3 are fulfilled, can be checked similarly. □

Theorem 4.2 *Let the assumptions of either of Theorems 2.1–2.3 be fulfilled and the following conditions hold*

(a') $f(., ., \lambda_0)$ *is* $\Gamma(., \lambda_0)$*-pseudomonotone in* $K(A, \lambda_0) \times K(A, \lambda_0)$;
(b') *if* $f(x, y, \lambda_0) \in \mathrm{bd}\Gamma(x, \lambda_0)$ *then* $x = y$;
(c') $f(x, \bar{x}, \lambda_0) \in \Gamma(x, \lambda_0)$ *for all* $x, \bar{x}$ *in* $S(\lambda_0)$.

Then S *is continuous at* λ_0.

Proof We retain the first part of the proof of Theorem 4.1, including a contradiction argument with the end that $\bar{x}_0 \neq x_0$. (c') implies that $f(x_0, \bar{x}_0, \lambda_0) \in \Gamma(x_0, \lambda_0)$ and $f(\bar{x}_0, x_0, \lambda_0) \in \Gamma(\bar{x}_0, \lambda_0)$. By (a$'$), one has $f(\bar{x}_0, x_0, \lambda_0) \in Y \backslash \mathrm{int}\Gamma(\bar{x}_0, \lambda_0)$. Hence $f(\bar{x}_0, x_0, \lambda_0) \in$ bd $\Gamma(\bar{x}_0, \lambda_0)$. Assumption (b$'$) now yields a contradiction that $\bar{x}_0 = x_0$. □

5 Particular cases

Since equilibrium problems contain many problems as special cases, including variational inequalities, optimization problems, fixed-point and coincidence-point problems, complementarity problems, Nash equilibrium problems, etc, we can derive from the results of Sects. 2–4 consequences for such special cases. In this section we discuss only some corollaries for quasivariational inequalities and traffic network problems as examples.

5.1 Quasivariational inequalities

Let X, A, Λ, K be as in Sect. 1, X^* be the dual space of X and $T : X \times \Lambda \to X^*$. We consider the following parametric quasivariational inequality, for each $\lambda \in \Lambda$,
(QVI) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that, for all $y \in K(\bar{x}, \lambda)$,

$$\langle T(y, \lambda), y - \bar{x} \rangle \geq 0,$$

where $\langle ., . \rangle$ denotes the pairing between X and X^*.

To convert (QVI) to a special case of (QEP) set $Y = \mathbb{R}$, $\Gamma(x, \lambda) = \mathbb{R}_+$ and $f(x, y, \mu) = \langle T(y, \mu), y - x \rangle$. Consequently, the following result is immediate from Theorem 2.1.

Corollary 5.1 *Assume for* (QVI) *that*

(i) E *is usc at* λ_0, $E(\lambda_0)$ *is compact and* K *is lsc in* $K(A, \Lambda) \times K(A, \Lambda)$;
(ii) *the set* $\{(x, y) \in A \times A \mid \langle T(y, \lambda_0), y - x \rangle \geq 0\}$ *is closed in* $K(A, \Lambda) \times K(A, \Lambda)$;
(iii) *for all* x, y *in* $K(A, \Lambda)$*, the function* $\lambda \mapsto \langle T(y, \lambda), y - x \rangle$ *is* $(-\infty, 0)$*-usc at* λ_0.

Then the solution map S *is usc at* λ_0.

Remark 5.1 (i) By Theorem 2.2, Corollary 5.1 is still valid if we replace assumptions (ii) and (iii) by (iii') The set $\{(x, y, \lambda) \mid \langle T(y, \lambda), y - x\rangle \geq 0\}$ is closed in $K(A, \Lambda) \times K(A, \Lambda) \times \{\lambda_0\}$.

(ii) Corollary 5.1 together with the above Remark 5.1(i) include Theorems 2.2 and 2.3 of [17], Theorems 4.1 and 4.3 of [18].

(iii) Similarly, we can obtain direct corollaries of Theorems 3.1, 3.2 and these results are new for (QVI), as far as we know.

5.2 Traffic network problems

The notion of equilibrium flows for transportation network problems was introduced in Wardrop [22] together with a basic traffic network principle. Since then, traffic network problems have raised a great interest and have been much developed in both theory and methodology view points. Variational approaches to such traffic problems began with Smith [21], where it was proved that the Wardrop equilibrium can be expressed in terms of variational inequalities. In [1,3,4,6,11] Hölder continuity of solution maps to such parametric elastic traffic problems was considered. In this subsection, utilizing results of Sect. 2 we investigate continuity properties of solutions of the following elastic traffic problem, which was considered by many authors, see e.g., [7,10,17] and references therein.

Let N be the set of nodes, L be that of links (or arcs), $W = (W_1, \ldots, W_l)$ be the set of origin-destination pairs (O/D pairs in short). Assume that the pair W_j, $j = 1, \ldots, l$, is connected by a set P_j of paths and P_j contains $r_j \geq 1$ paths. Let $F = (F_1, \ldots, F_m)$ be the path vector flow, where $m = r_1 + \ldots + r_l$. Assume that the capacity restriction is

$$F \in A := \{F \in R^m : 0 \leq \gamma_s \leq F_s \leq \Gamma_s, s = 1, \ldots, m\},$$

where γ_s and Γ_s are given real numbers. Assume further that the travel cost on the path flow F_s, $s = 1, \ldots, m$, depends on the whole path vector flow F and is $T_s(F) \geq 0$. Then we have the path cost vector $T(F) = (T_1(F), \ldots, T_m(F))$.

Following Wardrop [22] a path vector flow H is said to be an equilibrium vector flow if $\forall W_j, \forall p \in P_j, \forall s \in P_j$,

$$[T_p(H) < T_s(H)] \Longrightarrow [H_s = \gamma_s \quad \text{or} \quad H_p = \Gamma_p].$$

Now assume that a perturbation on the traffic is expressed by parameter c of a metric space C. Assume further that a travel demand g_j of the O/D pair W_j depends on $c \in C$ and also on the equilibrium vector flow H. Denote $g = (g_1, \ldots, g_l)$ and set

$$\phi_{js} = \begin{cases} 1, & \text{if } \ s \in P_j, \\ 0, & \text{if } \ s \notin P_j, \end{cases}$$
$$\phi = \{\phi_{js}\}, j = 1, \ldots, l; s = 1, \ldots, m.$$

Then the path vector flows meeting the travel demands are called the feasible path vector flows and form the constraint set

$$K(H, c) = \{F \in A \mid \phi F = g(H, c)\}.$$

ϕ is called an O/D pair-path incidence matrix. Assume further that the path costs are also perturbed, i.e., depend on a perturbation parameter b of a metric space B: $T_s(F, b)$, $s = 1, \ldots, m$. Note that the "path model" (where the variables are path flows) we use here does not need the additivity of the travel cost as in so-called arc models.

Our traffic network problem is equivalent to a quasivariational inequality as follows.

Lemma 5.2 (See [15,21]) *A path vector flow $H \in K(H, a)$ is an equilibrium flow if and only if it is a solution of the following quasivariational inequality*

(TNP) *Find $H \in K(H, c)$ such that, for all $F \in K(H, c)$,*

$$\langle T(H, b), F - H\rangle \geq 0.$$

We need the following simple assertions

Lemma 5.3 (See [1, Lemma 1]) *Let A be an $m \times n$ matrix, a_1 and a_2 be given vectors in R^m. The solution set of the linear equality $Ax = a_i$, for i = 1, 2, is denoted by S_i. Then, there exists $\delta = \delta(A) > 0$ such that for each $x_1 \in S_1$ there exists $x_2 \in S_2$ satisfying*

$$\|x_1 - x_2\| \leq \delta\|a_1 - a_2\|.$$

Lemma 5.4 *Assume that g is continuous at (H_0, c_0). Then K is continuous at (H_0, c_0) and convex, compact-valued.*

Proof Consider the system

$$\phi F = g(H_0, c_0),$$
$$\phi F = g(H, c).$$

By Lemma 5.3, there exists $\delta > 0$ such that, for each $F_0 \in K(H_0, c_0)$, there exists $F \in K(H, c)$ satisfying

$$\|F - F_0\| \leq \delta\|g(H, c) - g(H_0, c_0)\|.$$

Since g is continuous at (H_0, c_0), K is lsc at (H_0, c_0). Suppose K is not usc at (H_0, c_0), i.e., there are an open superset U of $K(H_0, c_0)$ and a sequence $\{(H_n, c_n)\}$ converging to (H_0, c_0) such that, for each n, there exists $F_n \in K(H_n, c_n)\backslash U$. By the compactness of A, we can assume that $F_n \to F_0$. According to Lemma 5.3, there is $F_n^0 \in K(H_0, c_0)$ such that

$$\|F_n - F_n^0\| \leq \delta\|g(H_n, c_n) - g(H_0, c_0)\|.$$

Consequently, $F_n^0 \to F_0$. Since $\phi F_n^0 = g(H_0, c_0)$, we have $\phi F_0 = g(H_0, c_0)$, i.e., $F_0 \in K(H_0, c_0) \subseteq U$, a contradiction. □

Setting $X = \mathbb{R}^m$, $\Lambda = C \times B$ and, for each $\lambda = (c, b) \in \Lambda$, $K_1(H, \lambda) = K(H, c)$ and $f(x, y, \lambda) = \langle T(x, b), y - x\rangle$. Then (TNP) becomes a special case of (QEP).

Corollary 5.5 *For problem* (TNP) *assume that*

(i) *g is continuous in $K(A, c_0) \times \{c_0\}$;*
(ii) *the set $\{(H, F, b) \mid \langle T(H, b), F - H\rangle \geq 0\}$ is closed in $A \times A \times \{b_0\}$.*

Then the solution map S is usc at (c_0, b_0).

Proof It is derived from Lemma 5.4 and Theorem 2.2. □

Corollary 5.6 *Impose the assumptions of Corollary 5.5 and the following conditions*

(a) *T is quasimonotone in $K(A, c_0)$;*
(b) *For each $H \in S(c_0, b_0)$ and each $H' \in S(c_0, b_0)\backslash\{H\}$, $\langle T(H, b_0), H' - H\rangle > 0$.*

Then S is continuous at (c_0, b_0).

Proof It is clear from Theorem 4.1. □

Corollary 5.7 *Let the assumptions of Corollary 5.5 and the following conditions be satisfied*

(a') T *is pseudomonotone in* $K(A, c_0)$*;*
(b') *for* H_1, H_2 *in* $E(\lambda_0)$*, if* $\langle T(H_1, b_0), H_2 - H_1\rangle = 0$ *then* $H_2 = H_1$;
(c') $\langle T(H_1, b_0), H_2 - H_1\rangle \geq 0$ *for all* H_1, H_2 *in* $S(c_0, b_0)$.

Proof It is a direct consequence of Theorem 4.2. □

Remark 5.2 Corollary 5.7 improves Theorem 4.1 of [20], since here (c′) needs to be fulfilled only on $S(c_0, b_0)$ and assumption (ii) is weaker than the continuity assumption of T required in this theorem. Corollaries 5.5 and 5.6 are new. We note further that the results in Sect. 5.1 can be applied for (TNP), but Theorems 3.1–3.3 in [18] cannot, since assumption (iii) in these theorems is not fulfilled in this case.

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